Analytic continuation of Eisenstein series

Thus far we have defined:

- $$E_s(g) = \sum_{\Gamma_B \backslash \Gamma} a(\gamma g)^{\rho + s} \qquad \left(\begin{array}{l} G = SL_n(\mathbb{R}) \text{ or } GL_n(\mathbb{R}) \\ s \in \mathbb{C}^n \end{array} \right.$$

  conv. abs. for $\mathrm{Re}(s) - \rho$ strictly dominant

- For each standard parabolic $P = MU$, $f \in C_c^\infty(\Gamma_P U \backslash G)$,
  $$\mathrm{Eis}(f)(g) := \sum_{\Gamma_P \backslash \Gamma} f(\gamma g).$$
  $$\rightsquigarrow \mathrm{Eis}(f) \in C_c^\infty(\Gamma \backslash G)$$

Formally, taking $P = B$, $f(uak) = a^{\rho+s}$,
we have $\mathrm{Eis}(f) = E_s$.

<u>Goal</u> Want to define $\mathrm{Eis}(f)$ for "any" automorphic form f on $\Gamma_P U \backslash G$.

<u>Issue</u> The series does not in general converge absolutely. Instead, meromorphically continue:

Let $P = MU$: standard parabolic $\qquad M = M_P$, $U = U_P$

Thus $M \cong GL_{n_1}(\mathbb{R}) \times \cdots \times GL_{n_r}(\mathbb{R}) \quad$ (for $G = GL_n(\mathbb{R})$)

where $n_1 + \dots + n_r = n$.

Set $X_P :=$ group of characters of M of the form
$$M \ni g \mapsto \prod_{j=1}^{r} |\det g_j|^{s_j}$$
$$g \Updownarrow (g_1, \dots, g_r), \quad g_j \in GL_{n_j}(\mathbb{R})$$

for some $s = (s_1, \dots, s_r) \in \mathbb{C}^r \cong X_P$.

Let $A_P := (\text{center of } M) \cong (\mathbb{R}^\times)^r$,
$\mathfrak{a}_P := \mathrm{Lie}(A_P)$. Then $X_P \cong \mathfrak{a}_{P,\mathbb{C}}^* := \mathrm{Hom}(\mathfrak{a}_P, \mathbb{C})$
$$\chi \mapsto d(\chi|_{A_P})$$

$G = BK = PK = UMK$

$g = u \cdot m_P(g) \cdot k$ up to $M \cap K$

$X_P \ni s \rightsquigarrow m_P(g)^s \in \mathbb{C}^\times$

NB $m_P(g)$ ambiguous, but $m_P(g)^s$: well-defined b/c $x^s = 1$ $\forall x \in M \cap K$.

clf f: automorphic form on $\Gamma_P U \backslash G$, then so is $f_s := f \cdot m_P^s$.

Lemma Suppose that s is "sufficiently dominant": $s_j - s_{j+1}$ is large enough in terms of the automorphic form f.

Then $\mathrm{Eis}(f_s)(g) = \sum_{\Gamma_P \backslash \Gamma} f_s(\gamma g)$ conv. abs.

Ex $P = B$, $f = 1$, then $\mathrm{Eis}(f_s) = E_s$.

(For the general case, see Borel, Aut. forms on reductive gps, final section.)

Theorem (Selberg, Langlands, see Bernstein-Lapid) (see also Iwaniec)

Let $P = MU$: std parabolic. f: aut. form on $\Gamma_P U \backslash G$.

Then $s \mapsto \mathrm{Eis}(f_s)$, defined initially for s: "sufficiently dom.", extends meromorphically to all of X_P.

Further properties include a functional equation, control over poles.

ex $E_s^* := \left(\prod_{i<j} \xi(1 + s_i - s_j) \right) E_s$

$(P = B, f = 1)$ | $\xi(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$

is invariant under permutations of $s = (s_1, \dots, s_n)$.

Defn $\mathrm{Eis}(f) := \mathrm{Eis}(f_s)|_{s=0}$.

Remark For $n=2$, $\Gamma = SL_2(\mathbb{Z})$ (or more generally, $\Gamma = SL_n(\mathbb{Z})$, $P = B$) one can give "direct proofs" using Fourier expansions OR Poisson summation.

$\underline{n=2}$ $$E_s(g) = \sum_{\Gamma_B \backslash \Gamma} a(\gamma g)^{s+\rho} \qquad a(\gamma g) = \begin{pmatrix} y & \\ & y^{-1} \end{pmatrix}$$

$\Gamma = SL_2(\mathbb{Z})$ $$X_B \leftrightarrow \mathbb{C} : \begin{pmatrix} y & \\ & y^{-1} \end{pmatrix}^s = y^s$$

$\Gamma_B = \begin{pmatrix} * & * \\ & * \end{pmatrix}$ $$\rho \leftrightarrow 1$$

$$\Gamma_B \backslash \Gamma \leftrightarrow \{1\} \amalg \left\{ \underbrace{\pm \begin{pmatrix} * & * \\ c & d \end{pmatrix}}_{\Gamma_B\text{- coset}} : c \geq 1,\ d \in \mathbb{Z},\ \gcd(c,d)=1 \right\}$$

$$g = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} k, \quad \text{so} \quad gK \leftrightarrow z \in \mathbb{H} \in G/K$$

$$E_s(g) = y^{\frac{1+s}{2}} + \sum_{c=1}^{\infty} \sum_{\substack{d \in \mathbb{Z} \\ \gcd(c,d)=1}} \underbrace{a\left(\begin{pmatrix} * & * \\ c & d \end{pmatrix} g \right)^{s+\rho}}$$

$$= \frac{y^{\frac{1-s}{2}}}{|cz+d|^{1+s}}$$

$$= \sum_{m \in \mathbb{Z}} e(mx)\, \underbrace{a_{m,s}(y)} + y^{\frac{1+s}{2}}$$

$$:= \int_{x \in \mathbb{R}/\mathbb{Z}} e(-mx) \sum_{c=1}^{\infty} \sum_{\substack{d \in \mathbb{Z} \\ \gcd(c,d)=1}} \frac{y^{\frac{1-s}{2}}}{|cz+d|^{1+s}}\, dx$$

$(z := x+iy)$

$$= y^{\frac{1-s}{2}} \sum_{c=1}^{\infty} \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^*} \underbrace{\int_{x \in \mathbb{R}} \frac{e(-mx)}{|cz+d|^{1+s}}\, dx}$$

$$\underset{\substack{\uparrow \\ x \mapsto x - d/c}}{=} e\left(\frac{md}{c}\right) \int \frac{e(-mx)}{|cz|^{1+s}}\, dx$$

Exercise: continue this calculation to deduce that

$$E_s(g) = \underbrace{y^{\frac{1+s}{2}} + y^{\frac{1-s}{2}} \frac{\xi(s)}{\xi(s+1)}}_{a_{0,s}(y)} + \underbrace{\frac{1}{\xi(s+1)} \sum_{m \neq 0} \frac{\tau_{s/2}(m)}{|m|^{1/2}} W_{s/2}(my) e(mx)}_{\text{entire in } s, \text{ inv. under } s \mapsto -s}$$

$(g \Leftrightarrow z)$

where $\tau_{s/2}(m) = \sum_{\substack{ab = |m| \\ a, b \in \mathbb{Z}_{\geq 1}}} (a/b)^{s/2}$,

$$W_{s/2}(y) = 2|y|^{1/2} K_{s/2}(2\pi|y|).$$

In this case, merom. cont. of E_s $\Leftrightarrow$ that of $\xi(s)$.
$\Leftrightarrow$ $\not\!\!\!\longrightarrow$ $\xi(s)$

functional equation $E_s^* := \xi(1+s) E_s = E_{-s}^*$
$\Leftrightarrow$ $\xi(s) = \xi(1-s)$.

Direct approach via Poisson summation:

Take $n=2$.

The series $F_t(g) := \sum_{v \in \mathbb{Z}^2 - \{0\}} \exp(-\pi \|tvg\|^2)$: decays rapidly as $t \to \infty$,

$(g \in Sl_2(\mathbb{R}), t > 0)$ (v: row vectors) $= -1 + t^{-2} + O(t^N)$ as $t \to 0$ $(\forall N)$.

conv. abs., functional equation:

$$t(1 + F_t(g)) \overset{\text{Poisson summation}}{=} t^{-1}(1 + F_{t^{-1}}(g)) \Rightarrow I(s) = I(-s)$$

The integral $I(s) := \int_0^\infty F_t(g)\, t^{1+s} \frac{dt}{t}$ conv. abs. for $Re(s) > 1$,
meromorphically continues to $\mathbb{C}$, simple poles at $s = \pm 1$, residues 1.
On the other hand, for $Re(s) > 1$, we may exchange Σ / S:

$$I(s) = \int_0^\infty \left(\sum_{\mathbb{Z}^2 - \{0\}} \exp(\text{--}) \right) t^{1+s} \frac{dt}{t}$$

$$F_t(g) = \sum_{\substack{v \in \mathbb{Z}^2/\pm 1 \\ \text{primitive}}} \sum_{n \in \mathbb{Z} - \{0\}} \exp\left(-\pi \|\underset{\sim}{tnv}\, g\|^2\right)$$

$$\Rightarrow I(s) = \sum_{\substack{v \in \mathbb{Z}^2/\pm 1 \\ \text{primitive}}} \sum_{n \in \mathbb{Z} - \{0\}} \underbrace{\int t^{1+s} \exp\left(-\pi \|tnvg\|^2\right) \frac{dt}{t}}$$

$$= \underbrace{n^{-1-s} \|vg\|^{-1-s} \int t^{1+s} \exp(-\pi t^2) \frac{dt}{t}}$$

$$= \|vg\|^{-1-s}\, \xi(1+s)$$

$$= 2^{(?)} \xi(1+s) \underbrace{E_s(g)}_{= 2^{(?)} \sum_{\substack{v \in \mathbb{Z}^2/\pm 1 \\ \text{prim}}} \|vg\|^{-1-s}}$$

$$\|vg\|^{-1-s} = \sum_{\substack{\gamma \in \Gamma_B \backslash \Gamma \\ \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}}} a(\gamma g)^{\rho + s}$$

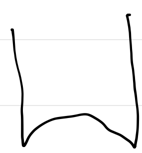
$$v = (c, d)$$

Generalizes to Eisenstein series for $P = B$.
(Gelfand - Graev - PS, Braverman - Kazhdan "Schwartz space")

We now turn to the general argument. (Borel, SL_2, §11.)
This applies more generally to "non-arithmetic" $\Gamma < G$.

ex $\quad G = SL_2(\mathbb{R}) > \Gamma$: cofinite

We'll focus on the case that Γ has one cusp.



We focus first on $G = SL_2(\mathbb{R})$, $P = \begin{pmatrix} * & * \\ & * \end{pmatrix}$, $M = \begin{pmatrix} * & \\ & * \end{pmatrix}$

We are given an aut. form f on $\Gamma_P U \backslash G$. $U = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$

We assume that

$$f_s\left(\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \begin{pmatrix} y & \\ & y^{-1} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right) = y^{1+s} e(m\theta)$$

for some $s \in \mathbb{C}$, $m \in \mathbb{Z}$.

Notation $E_s := E_{is}(f_s)$: conv. abs. for $\mathrm{Re}(s) > 1$.

Lemma $E_{s,P} = f_s + c(s) f_{-s}$ for some $c(s) \in \mathbb{C}$.

Proof Bruhat decomposition of G: $\left(w = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \right)$

$$G = B \amalg BwU \qquad B \times U \to BwU \text{ is bijective}$$

$$B \backslash G = \{1\} \amalg wU$$

$$\Gamma_B \backslash \Gamma = \{1\} \amalg \Gamma_B \backslash \Gamma_w, \qquad \Gamma_w := \Gamma \cap BwU$$

$$BwU = \left\{ \begin{pmatrix} * & * \\ c & * \end{pmatrix} : c \neq 0 \right\}$$

$$\Rightarrow E_s(g) = \sum_{\Gamma_B \backslash \Gamma} f_s(\gamma g) = f_s(g) + \underbrace{\sum_{\gamma \in \Gamma_B \backslash \Gamma_w} f_s(\gamma g)}_{=: E_{s,w}(g)}$$

$$E_{s,w}(g) = \sum_{\gamma \in \Gamma_B \backslash \Gamma_w / \Gamma_U} \sum_{\delta \in \Gamma_U} f_s(\gamma \delta g)$$

$$E_{s,P}(g) = \int_{u \in \Gamma_U \backslash U} E_s(ug)\, du = f_s(g) + E_{s,w,P}(g)$$

$$E_{s,w,P}(g) = \sum_{\gamma \in \Gamma_B \backslash \Gamma_w / \Gamma_U} \underbrace{\int_{u \in \Gamma_U \backslash U} \sum_{\delta \in \Gamma_U} f_s(\gamma \delta u g)\, du}_{= \int_{u \in U} f_s(\gamma u g)\, du}$$

Claim $\forall a \in A$, $\int_U f_s(\gamma u a g)\, du = a^{-s+\rho} \int_U f_s(\gamma u g)\, du$.

$\Rightarrow$ Lemma b/c f_{-s} is the unique (up to scalar) function transforming on the left under B via $\begin{pmatrix} y & * \\ & y^{-1} \end{pmatrix} \mapsto y^{1-s}$, right under K by $e(m\theta)$.